Extremal unital completely positive normal maps

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Date: 12th Jan, 2013

Abstract:

We study the convex set of unital completely positive normal map on a von-Neumann algebra and find a necessary and sufficient condition for an element in the convex set to be extremal. We also deal with the same problem for the convex subset which admits a faithful normal state.

1 Introduction:

Let $\mathcal{M} = M_n(\mathbb{C})$ be the matrix algebra over the field of complex numbers. Any completely positive unital map τ [St] on $\mathcal{M}$ admits Stinespring's minimal representation $\tau(x) = \sum_k v_k x v_k^*$ where $\{v_k \in \mathcal{M} : 1 \leq k \leq d\}$ are linearly independent. It is quite some time now that M. D. Choi's described a useful criteria for an element τ to be extremal in the convex set CP of completely positive unital map. The elegant criteria [Ch] says τ is an extremal element if and only if the family $\{v_k v_j^* : 1 \leq k, j \leq d\}$ is linearly independent. Extreme points in CP is a building block to test various physical ansatz in quantum information theory [MW].

The proof that M.D. Choi executed, uses one crucial observation that the linear space $\mathcal{L}_\tau$ generated by elements $\{v_k^* : 1 \leq k \leq d\}$ is independent of the representation that one may choose to represent τ. Same problem with $\mathcal{M}$, an arbitrary von-Neumann algebra or more generally a C^*-algebra remains open as an interesting problem till then. It is not hard to see that Choi's method can not be adapted to the general situation. On the other hand Arveson's affine correspondence [Ar] between the CP maps and states on $M_d(M_n(C))$ has little use in the general set up as correspondence uses matrix basis explicitly for $\mathcal{M}$.

In this paper we invent an independent method largely generalizing Dixmiér version of Radon-Nykodym theorem for two normal states to two normal completely positive maps. Main comparable mathematical result with that of Choi's result says that if τ, η are two unital completely positive normal maps on $\mathcal{M}$ and $\eta \leq k\tau$ on $\mathcal{M}_+$, non-negative elements of $\mathcal{M}$ for some constant $k > 0$ then

$$\eta(x) = \sum_k v_k x t_j^k v_j^*$$

where $\tau(x) = \sum_k v_k x v_k^*$ is the minimal representation of Stinespring and $t_j^k \in \mathcal{M}'$ are elements determined uniquely with $t = (t_j^k)$ non-negative. Main application of this result says that τ is extremal if and only if there exists no non-trivial solution

with $\lambda = (\lambda^j_k)$ with entries in $\mathcal{M}'$ to

$$\sum v_k \lambda^k_j v^*_j = 0$$

In case τ admits an inner representation i.e. with $v_k \in \mathcal{M}$ for all k, then same holds with λ^j_k taking values in the center of $\mathcal{M}$. Thus our main result in particular captures classical result of Choi [Ch].

In section 3 we investigate the same problem of finding criteria for extremal property of an element τ in CP_ϕ, the convex set of unital completely positive map on $\mathcal{M}$ with a faithful normal invariant state ϕ on $\mathcal{M}$. Here we took some hint from work of [Oh] which followed work of [Pa] and formulate the problem in the framework of Tomita's coupling of $\mathcal{M}$ and $\mathcal{M}'$. Method here is as well striking different from Landau-Streater's adaptation of Choi's method.

2 Extremal decomposition of a completely positive unital map:

Let $\mathcal{A}$ be a unital C^*-algebra. A linear map $\tau : \mathcal{A} \to \mathcal{A}$ is called positive if $\tau(x) \geq 0$ for all $x \geq 0$. Such a map is automatically bounded with norm $||\tau|| = ||\tau(I)||$. Such a map τ is called completely positive [St] (CP) if $\tau \otimes I_n : \mathcal{A} \otimes M_n \to \mathcal{A} \otimes M_n$ is positive for each $n \geq 1$ where $\tau \otimes I_n$ is defined by $(x^i_j) \to (\tau(x^i_j))$ with matrix entries (x^i_j) are elements in $\mathcal{A}$. In this paper we will only consider unital completely positive maps i.e. $\tau(I) = I$. We denote by $CP(\mathcal{A})$ or simply CP for the convex set of unital completely positive map on $\mathcal{A}$. Two elements τ, τ' are said to be (anti) cocycle conjugate if $\tau \circ \alpha = \beta \circ \tau'$ for two (anti-) automorphisms α, β on $\mathcal{A}$. They are called (anti-) conjugate if $\tau \circ \alpha = \alpha \circ \tau'$ for an (anti-) automorphism α.

An unital completely positive map on an unital C^* algebra $\mathcal{A}$ admits a minimal Stinespring representation [St]

$$\tau(x) = V \pi(x) V^*$$

where $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}^s)$ is a representation of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H}^s)$ which is the Hilbert space completion of the kernel given on the set $\mathcal{A} \otimes \mathcal{H}$ by

$$k(x \otimes \zeta, y \otimes \eta) =< \zeta, \tau(x^* y)\eta >$$

and $V^* : \mathcal{H} \to \mathcal{H}^s$ is an isometry from $\mathcal{H}$ into $\mathcal{H}^s$ defined by $V^* : \zeta \to I \otimes \zeta$ such that $\{\pi(x)V^*\zeta : x \in \mathcal{A}, \zeta \in \mathcal{H}\}$ spans $\mathcal{H}^s$. Such a minimal Stinespring's triplet $(\mathcal{H}^s, \pi, V^*)$ is uniquely determined modulo unitary equivalence i.e. if $(\mathcal{H}^{s'}, \pi', V'^*)$ be another triplet associated with τ, then we get $U : \pi(x)V^*\zeta \to \pi'(x)V'^*\zeta$ is an unitary operator so that $UVU^* = V'$ and $U\pi(x)U^* = \pi'(x)$.

Let $\mathcal{M}$ be a von-Neumann algebra acting on a complex separable Hilbert space $(\mathcal{H}, < .,. >)$, where the inner product is linear in second variable and $\mathcal{M}_*$ be the pre-dual space of $\mathcal{M}$. We say τ is normal if $l.u.b.\tau(x_\alpha) = \tau(l.u.b.x_\alpha)$ for any bounded increasing net x_α in $\mathcal{M}$ where $l.u.b.$ denotes least upper bound. We will use notation $CP^\sigma(\mathcal{M})$ for the convex set of unital completely positive normal maps on $\mathcal{M}$. When there is no room for confusion we will simply denote it by CP^σ.

For a von-Neumann algebra $\mathcal{M}$, it is well known that norm closed unit ball of $\mathcal{M}_*$ is an open dense subset in the norm closed unit ball of $\mathcal{M}^*$ in the weak* topology on $\mathcal{M}^*$. Further $\mathcal{M}_*$ is a proper subset unless $\mathcal{M}$ is finite dimensional. However $\mathcal{M}_*$ is sequentially closed [Ta] and given any bounded linear functional ω on $\mathcal{M}$ there is a unique element $\omega_\sigma \in \mathcal{M}_*$ so that $\omega = \omega_\sigma + \omega_s$ determined by $||\omega - \omega_\sigma|| = \min_{\omega' \in \mathcal{M}_*} ||\omega - \omega'||$. The element ω_s is a singular functional unless it is the zero element. In the following we describe such a decomposition via universal enveloping algebra of $\mathcal{M}$.

Let $\mathcal{S}(\mathcal{A})$ be the convex set of states on $\mathcal{A}$. The universal enveloping algebra $\mathcal{A}^{**}$ of a C^*-algebra i.e. the double dual $\mathcal{A}^{**}$ of $\mathcal{A}$ is a von-Neumann algebra, unitary equivalent to von-Neumann algebra $\mathcal{M}_u^{\mathcal{A}} = \{\pi_u(x) : x \in \mathcal{A}\}''$ where $\pi_u(x) = \oplus_{\phi \in \mathcal{S}(\mathcal{A})} \pi_\phi(x)$ is the direct sum of representations on $\mathcal{H}_u = \oplus_{\phi \in \mathcal{S}(\mathcal{A})} \mathcal{H}_\phi$ and $(\mathcal{H}_\phi, \pi_\phi, \zeta_\phi)$ is the GNS representation associated with a state ϕ on $\mathcal{A}$. In the

following text we identify $\mathcal{A}^{**}$ with $\mathcal{M}_u^{\mathcal{A}}$. Let $i : \mathcal{A} \to \mathcal{A}^{**} \equiv \mathcal{M}_u^{\mathcal{A}}$ be the inclusion map of $\mathcal{A}$ into $\mathcal{A}^{**}$.

For a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi)$, we set von-Neumann algebra $\mathcal{M}_\pi = \pi(\mathcal{A})''$ and the Banach space adjoint linear map by $\pi^t : \mathcal{M}_\pi^* \to \mathcal{A}^*$. We set now linear map $(\pi^t)_* : (\mathcal{M}_\pi)_* \to \mathcal{A}^*$ by restricting π^t to $(\mathcal{M}_\pi)_*$. Finally we set notation $\tilde{\pi} : \mathcal{A}^{**} \to \mathcal{M}_\pi$ for the Banach space adjoint map of $(\pi^t)_*$. Similarly for a completely positive map $\tau : \mathcal{A} \to \mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ where $\mathcal{M}$ is a von-Neumann algebra acting on a Hilbert space $\mathcal{H}$, we set $\tilde{\tau} : \mathcal{A}^{**} \to \mathcal{M}$ for adjoint map of $(\tau^t)_* : \mathcal{M}_* \to \mathcal{A}^*$.

PROPOSITION 2.1: We have the following property of universal von-Neumann algebra $\mathcal{A}^{**}$:

(a) $\tilde{\pi}$ is a linear map which is continuous with respect to weak topologies of $\mathcal{A}^{**}$. The map $\tilde{\pi}$ takes norm closed unit ball of $\mathcal{A}^{**}$ onto the closed unit ball of $\mathcal{M}_\pi$;

(b) $\tilde{\pi} \circ i = \pi$ on $\mathcal{A}$;

(c) $\tilde{\pi}$ is an unital completely positive map from $\mathcal{A}^{**}$ onto $\mathcal{M}_\pi$;

(d) For any central element $z \in \mathcal{A}^{**}$, $\tilde{\pi}(z)$ is an element in the center of $\mathcal{M}_\pi$.

PROOF: Statement (a) and (b) are well known property of universal enveloping algebra of a C^*-algebra for which we refer to Lemma 2.2 in Chapter 3 of [Ta].

Statement (c) and (d) are as well known but could not find a suitable ready reference. Here we indicate a proof. Since π is a positive map and so is it's transpose π^t. Thus the restriction $(\pi^t)_*$is also positive. That $\tilde{\pi}$ is positive follows as $(\pi^t)_*$ is positive and onto. For $n-$positive property of $\tilde{\pi}$, we note that the universal enveloping algebra over $M_n(\mathcal{A})$ is $M_n(\mathcal{A}^{**})$ and the canonical map $i \otimes I_n$ is the inclusion map of $M_n(\mathcal{A})$ into $M_n(\mathcal{A}^{**})$. Further for a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi)$, we also have $\tilde{\pi} \otimes I_n \circ i \otimes I_n = \pi \otimes I_n$. This shows in particular that $\tilde{\pi}$ is a completely positive map and it's restriction on $i(\mathcal{A})$ is a representation.

By Kadison-Schwarz inequality for unital completely positive map we have $\tilde{\pi}(x^* y) = \tilde{\pi}(x^*)\tilde{\pi}(y)$ for $x \in \mathcal{A}^{**}$ and $y \in i(\mathcal{A})$ since $\tilde{\pi}(y^* y) = \tilde{\pi}(y^*)\tilde{\pi}(y)$ for $y \in i(\mathcal{A})$.

Taking adjoint in the above relation we also get $\tilde{\pi}(y^*x) = \tilde{\pi}(y^*)\tilde{\pi}(x)$. Operator algebras involved are being $*$-closed and also $\tilde{\pi}$ being onto, we get $\tilde{\pi}(x) \in \mathcal{M}_\pi \bigcap \mathcal{M}'_\pi$ if $x \in \mathcal{A}^{**} \bigcap \mathcal{A}^{**'}$. ■

It is well known that a projection in the center of $\mathcal{A}^{**}$ determines uniquely a representation of $\mathcal{A}$ upto quasi-equivalence i.e. a representation π is quasi-equivalent to the sub-representation $x \to \pi_u(x)z_\pi$ for some central projection z_π. z_π is called support of the representation in $\mathcal{A}^{**}$. For details we once more refer to Chapter 3 in [Ta].

Now we take $\mathcal{A} = \mathcal{M}$, a von-Neumann algebra in Proposition 2.1 and $\pi : \mathcal{M} \to \mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be the identity representation. Let z_π be the support projection of the representation $\pi : \mathcal{M} \to \mathcal{M}$ in $\mathcal{M}^{**}$ [Ta, Chapter 3]. Thus $\tilde{\pi}(z_\pi)$ is an element in the center of $\mathcal{M}$, where $\tilde{\pi}$ is the lift of π to the universal enveloping algebra defined in Proposition 2.1.

We defined action of a C^*-algebra $\mathcal{A}$ on it's dual $\mathcal{A}^*$ by

$$< y, x\omega >=< yx, \omega > \quad < y, \omega x >=< xy, \omega >$$

where $< .,. >$ denotes evaluation of a functional on a given element in the Banach space. A subspace of $\mathcal{A}^*$ is called $\mathcal{A}$ invariant if the subspace is invariant by both left and right action. The crucial property used to define action that $\mathcal{A}$ is algebra.

PROPOSITION 2.2: Let $\mathcal{M}$ be a von-Neumann algebra with it's pre-dual $\mathcal{M}_*$. A closed $\mathcal{M}$-invariant subspace V of $\mathcal{M}_*$ is determined unique by a projection z in the center of $\mathcal{M}$ by $V = z\mathcal{M}_*$.

PROOF: We refer to Theorem 2.7 in chapter 3 in [Ta]. ■

PROPOSITION 2.3: Let $\mathcal{M}$ be a von-Neumann algebra with it's pre-dual space $\mathcal{M}_*$ and dual $\mathcal{M}^*$. Then there exist a unique central projection $z \in \mathcal{M}^{**}$, the universal enveloping von-Neumann algebra so that $\mathcal{M}_* = z\mathcal{M}^*$ where $\mathcal{M}^{**}$ acts

natural on the Banach space $\mathcal{M}^*$ given by

$$< y, x\omega >=< yx, \omega > \quad < y, \omega x >=< xy, \omega >$$

for $x, y \in \mathcal{M}^{**}$ and $\omega \in \mathcal{M}^*$

PROOF: It is a simple application of Proposition 2.2 once we take $\mathcal{M}$ in Proposition 2.2 as $\mathcal{M}^{**}$ so $\mathcal{M}_*$ in Proposition 2.2 is $\mathcal{M}^*$. ■

PROPOSITION 2.4: An element $\omega \in \mathcal{M}^*$ determines uniquely an element $\omega_\sigma \in \mathcal{M}_*$ defined by

$$\omega_\sigma(x) =< \omega, zi(x) >$$

for all $x \in \mathcal{M}$ and the map $\omega \to \omega_\sigma \in \mathcal{M}^*$ is positive linear contractive map on the Banach space $\mathcal{M}_*$. The element $\omega_s(x) =< \omega, (1-z)i(x) >$ in $\mathcal{M}^*$ is singular provided $z \neq 1$. The decomposition $\omega = \omega_\sigma + \omega_s$ is also uniquely determined and $||\omega|| = ||\omega_\sigma|| + ||\omega_s||$.

PROOF: We refer chapter 3 in [Ta1] for details.

The following proposition says along the same vein now for the class of completely positive maps on $\mathcal{M}$.

THEOREM 2.5: Given a completely positive map τ on $\mathcal{M}$, there is a unique normal completely positive map τ_σ on $\mathcal{M}$ and a singular completely positive map τ_s such that $\tau = \tau_\sigma + \tau_s$ where τ_σ, τ_s are determined uniquely by the decomposition $\omega\tau_\sigma = (\omega\tau)_\sigma$, $\omega\tau_s = (\omega\tau)_s$ of a normal state ω. Further the set of normal completely positive maps on $\mathcal{M}$ is sequentially closed in the set of completely positive maps on $\mathcal{M}$ and open dense set in CP with Bounded Weak topology.

PROOF: Let $\mathcal{M}^{**}$ be the universal von-Neumann algebra of $\mathcal{M}$ and $i : \mathcal{M} \to \mathcal{M}^{**}$ be the canonical inclusion map of $\mathcal{M}$ into $\mathcal{M}^{**}$. We write for an element $\omega \in \mathcal{M}_*$

$$(\omega\tau)_\sigma(x) =< \omega\tau, zi(x) >$$

where z is the central projection in $\mathcal{M}^{**}$. The map $\omega \to (\omega\tau)_\sigma$ determines a normal positive linear map τ^σ on $\mathcal{M}$ by $\omega\tau_\sigma(x) =< \omega\tau, zi(x) >$ for all $x \in \mathcal{M}$ and $\omega \in \mathcal{M}_*$. It also shows clearly that τ_σ is completely positive as $\tau_\sigma \otimes I_n = (\tau \otimes I_n)_\sigma$. Similarly we also have $\omega\tau_s(x) =< \omega\tau, (1-z)i(x) >$ for all $x \in \mathcal{M}$, $\omega \in \mathcal{M}_*$ and the induced map τ_s is also completely positive.

Let τ_n be a sequence of such normal maps on $\mathcal{M}$ and it's bounded weak limit is τ i.e. $\omega(\tau_n(x)) \to \omega(\tau(x))$ for all $\omega \in \mathcal{M}_*$ and $x \in \mathcal{M}$. Let τ_s be the singular part of τ. Then singular part of $\omega\tau_n$ also converges to singular part of $\omega\tau$ in weak* topology. Since each τ_n is normal and so is ω, we get $\omega\tau_s^n = 0$. Since this holds for all $\omega \in \mathcal{M}_*$, we arrive at our desired claim that $\tau_s = 0$.

We will show complement of CP_σ is a closed set in BW topology. Let τ_α be a net of completely positive singular map on $\mathcal{M}$ converges to τ in Bounded Weak topology i.e. $\omega\tau_\alpha(x) \to \omega\tau(x)$ for all $x \in \mathcal{M}$ and $\omega \in \mathcal{M}_*$. Let $\tau = \tau_\sigma + \tau_s$ be the unique decomposition. Thus $\omega\tau_\alpha \to \omega\tau_s + \omega\tau_\sigma$. Note that each element $\omega\tau_\alpha$ is a singular functional i.e. an element in $\mathcal{M}_*^\perp$ and which is closed. Thus limiting element is also a functional in $\mathcal{M}_*^\perp$. This shows that $\omega\tau_\sigma = 0$ for all $\omega \in \mathcal{M}_*$.

A completely positive map is called *purely singular* in short singular if it's normal part of the unique decomposition contribute the zero map. We denote by CP_s for the convex set of unital singular completely positive maps on $\mathcal{M}$. One natural question that arises at this point when an unital completely positive map on $\mathcal{M}$ is a convex combination of unital normal and singular completely positive maps? For any unital representation π, $\pi_\sigma(I) = z$ and $\pi_s(I) = I - z$ and thus is a convex combination of two such unital map if and only if $z = 0$ or $z = I$. However we have the following observation. Furthermore any unital representation $\pi : \mathcal{M} \to \mathcal{M}$ is an extremal element in CP. Let $\pi = \lambda\tau_0 + (1-\lambda)\tau_1$ for some $\tau_0, \tau_1 \in CP$ and $\lambda \in (0,1)$. Then for a projection e, we check that $\pi(1-e)\tau_k(e)\pi(1-e) = 0$ for $k = 0, 1$ i.e. $\tau_k(e) \leq \pi(e)$ for all $k = 0, 1$ and $\lambda\tau_0(e) + (1-\lambda)\tau_1 = \pi(e)$. In particular $\pi_0(e)$ commutes with $\pi_1(e)$ and thus isomorphic to measurable functions $0 \leq f_0, f_1 \leq 1$ on

a measure space with $\lambda f_0 + (1-\lambda)f_1 = 1$. Since $\lambda \in (0,1)$ we have $f_0 = f_1 = 1$. Now pushing back via the isomorphism we get $\tau_0(e) = \tau_1(e) = \pi(e)$ for all projections e i.e. $\pi_0 = \pi_1 = \pi$. The proof would have been completed if π is also normal. For a general π we consider the lifting of π, π_0, π_1 to normal completely positive map from $\mathcal{M}^{**}$ to $\mathcal{M}$. Thus we have $\tilde{\pi} = \lambda\tilde{\pi}_0 + (1-\lambda)\tilde{\pi}_1$ on $\mathcal{M}^{**}$. Since $\tilde{\pi}$ is homomorphism on $i(\mathcal{M})$ and $\tilde{\pi} \circ i(x) = \pi(x)$, we conclude that $\pi_k(x) = \tilde{\pi}_k \circ i(x) = \tilde{\pi} \circ i(x) = \pi(x)$ for all $x \in \mathcal{M}$. Thus π is an extremal element in the set of positive unital map on $\mathcal{M}$ and so is in extremal element in CP.

PROPOSITION 2.6: Let $\mathcal{M}$ be von-Neumann algebra and z_0 be central projection in the universal enveloping algebra $\mathcal{M}^{**}$ such that $\mathcal{M}_* = z_0\mathcal{M}^*$. An element $\tau \in CP$ is a convex combination of two unital maps $\tau_\sigma \in CP_\sigma$ and $\tau_s \in CP_s$ with $\lambda \in (0,1)$ i.e.

$$\tau = \lambda\tau_\sigma + (1-\lambda)\tau_s$$

if and only if $\tilde{\tau}(z_0)$ is a scaler where $\tilde{\tau}$ is the unique completely positive unital normal map from $\mathcal{M}^{**}$ to $\mathcal{M}$ such that following diagram commutes:

$$< \omega\tau, x > = < \omega, \tilde{\tau}(x) > \tag{2.1}$$

for all $x \in \mathcal{M}^{**}$ and $\omega \in \mathcal{M}_*$. Further the map $\tau \to \tilde{\tau}$ is an affine one to on map.

PROOF: Let $\tau(x) = V\pi(x)V^*$ be the Stinespring's minimal representation of τ i.e. $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is unique unital representation in a Hilbert space $\mathcal{K}$ and $V^* : \mathcal{K} \to \mathcal{H}$ is the unique contraction (modulo unitary equivalence). It shows clearly

$$\tau_\sigma(x) = V\tilde{\pi}(z_\pi i(x))V^*$$

and

$$\tau_s(x) = V\tilde{\pi}(z_\pi i(x))V^*$$

where z_π is the support of π in $\mathcal{M}^{**}$ and $\tilde{\pi} : \mathcal{M}^{**} \to \mathcal{B}(\mathcal{K})$ be the lifting map of the representation $\pi : \mathcal{M} \to \mathcal{B}(\mathcal{K})$ defined as in Proposition 2.1. By Proposition 2.1 (d), projection $\tilde{\pi}(z_\pi)$ is an element in the center of $\mathcal{M}_\pi = \{\pi(x) : x \in \mathcal{M}\}''$.

Alternative description of τ_σ and τ_s are as follows. Let z_0 be the central projection in $\mathcal{M}^{**}$ such that $z_0\mathcal{M}^* = \mathcal{M}_*$. Then z_0 is the support projection of the identity representation $\pi_0 : \mathcal{M} \to \mathcal{M}$ i.e. $\pi_0(x) = x$ for all $x \in \mathcal{M}$ which is quasi-equivalent to $i_0 : \mathcal{M} \to \pi_u(\mathcal{M})z$. So $(\pi^t)_* : \mathcal{M}_* \to \mathcal{M}^*$ is the inclusion map and thus we have $\omega\tilde{\pi}_0 \circ i = \omega$ as an element in $\mathcal{M}^*$ for any normal state ω on $\mathcal{M}$.

We recall now

$$(\omega\tau)_\sigma(x) =< \omega\tau, z_0 i(x) >=< \omega, \tilde{\tau}(z_0 i(x)) >$$

for all $x \in \mathcal{M}$ and $\omega \in \mathcal{M}_*$. Uniqueness of the decomposition into normal and singular completely positive maps ensures that

$$\tau_\sigma(x) = \tilde{\tau}(z_0 i(x))$$

So in particular we have $\tilde{\tau}(z_0) = V\tilde{\pi}(z_\pi)V^* \in \mathcal{M}$. That $\tilde{\tau}(z_0)$ is a scaler if and only if τ is a convex combination of two unital maps from CP_σ and CP_s respectively with $\lambda = \tilde{\tau}(z_0)$ follows from uniqueness of the decomposition. ■

PROPOSITION 2.7: Let τ be a completely positive map on a C^* algebra $\mathcal{A}$ and η be another completely positive map on $\mathcal{A}$ so that for some positive constant c, $\eta(x) \leq c\tau(x)$ for all $x \in \mathcal{A}_+$. Then there exists a non-negative element $T' \in \pi(\mathcal{A})'$ such that

$$\eta(x) = V\pi(x)T'V^* \tag{2.2}$$

where $(\mathcal{K}, \pi, V^*)$ is the minimal Stinespring triplet associated with τ. Further for each $x \in \mathcal{M}$, $\pi(x)$ and T' commutes with the representation $\rho : \tau(\mathcal{A})' \to \mathcal{B}(\mathcal{K})$ defined by

$$\rho(y) : x \otimes f = x \otimes yf \tag{2.3}$$

PROOF: Without loss of generality we assume that $\mathcal{A}$ is a closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, $\mathcal{H}$ be a Hilbert space. We recall Stinespring minimal representation $\tau(x) = V\pi(x)V^*$ where $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is the representation induced by the map $y \otimes f \to$

$xy \otimes f$ where $\mathcal{H}$ is von-Neumann's completion of algebraic tensor product $\mathcal{A} \otimes \mathcal{H}$ with the kernel

$$k(x \otimes f, y \otimes g) =< f, \tau(x^*y)g >$$

Now we set another sesqui-linear form on $\mathcal{A} \otimes \mathcal{H}$ defined by

$$s(x \otimes f, y \otimes g) =< f, \eta(x^*y)g >$$

Being a kernel, we have Cauchy-Schwarz inequality to check the following steps:

$$|s(x \otimes f, y \otimes g)|^2 \leq s(x \otimes f, x \otimes f)s(y \otimes g, y \otimes g) \leq c^2||x \otimes f||||y \otimes g||$$

So there exists a bounder operator T' on $\mathcal{K}$ such that $s(x \otimes f, y \otimes g) =< x \otimes f, T'y \otimes g >$. We claim that $T' \in \pi(\mathcal{A})'$. Proof follows once check the following steps:

$$< f, \eta(x^*zy)g >= s(x \otimes f, zy \otimes g >=< x \otimes f, T'\pi(z)y \otimes g >$$

and

$$< f, \eta(x^*zy)g >= s(z^*x \otimes f, y \otimes g >=< \pi(z^*)x \otimes f, T'y \otimes g >$$

This completes the proof. ∎

THEOREM 2.8: Let $\mathcal{A}$ be a C^*-algebra and $\tau : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ be an unital completely positive map and $\tau(x) = V\pi(x)V^*$ be the unique upto isomorphism minimal Stinespring representation. Then following are equivalent

(a) τ is extremal in CP;

(b) $V\Lambda V^* = 0$ for $\Lambda \in \pi(\mathcal{A})'$ if and only if $\Lambda = 0$;

PROOF: Let $\tau = \lambda\tau_1 + (1-\lambda)\tau_0$ for some $\tau_0, \tau_1 \in CP$ and $\lambda \in (0,1)$. By Proposition 2.7 we have $\tau_0(x) = V\pi(x)T'V^*$ for some non-negative $T' \in \pi(\mathcal{A})'$. τ_0 being unital we also have $VT'V^* = I$ i.e. $V(I - T')V^* = 0$. In case (b) is true then $T' = I$ and thus τ_0 and so $\tau_1 = \tau_0 = \tau$. This proves (b) implies (a). For the converse $\Lambda \in \pi(\mathcal{A})'$ such that $V\Lambda V^* = 0$. Same holds if we replace symmetric or anti-symmetric part of Λ. Thus it enough if prove (b) for self-adjoint Λ. Since it is a bounded operator, we assume without loss of generality that $-I \leq \Lambda \leq I$. So both

$0 \le I - \Lambda \le 2I$ and $0 \le I + \Lambda \le 2I$ are operators in $\pi(\mathcal{A})'$. We write $I - \Lambda = W^*W$ and $I - \Lambda = W'^*W'$ for some W, W' in $\pi(\mathcal{A})'$ and check that $\tau = \frac{1}{2}(\tau_W + \tau_{W'}$ where $\tau_W = VW\pi(x)W^*V^*$ and $\tau_{W'} = VW'\pi(x)W'^*V^*$ Since τ is extremal, we conclude that $\tau_W = \tau_{W'} = \tau$. Thus $(\mathcal{K}, \pi, V^*)$ and $(\mathcal{K}, \pi, W^*V^*)$ are two Stinespring minimum triplet. Thus $W^*V^* = U^*V^*$ and $U\pi(x)U^* = \pi(x)$ for all $x \in \mathcal{A}$ where U is an unitary operator on $\mathcal{K}$. So we get $(UW^* - I)V^* = 0$. Since both U, W commutes with $\pi(\mathcal{A})$, we get $(UW^* - I)\pi(x)V^*f = 0$ for all $f \in \mathcal{H}$ and $x \in \mathcal{A}$. By cyclic property $\{\pi(x)V^*f : x \in \mathcal{A}, f \in \mathcal{H}\}$ is total in $\mathcal{K}$, thus $UW^* - I = 0$ i.e. W is unitary. So $\Lambda = I - W^*W = 0$. ∎

COROLLARY 2.9: Let $\tau : \mathcal{M} \to \mathcal{B}(\mathcal{H})$ be an unital normal completely positive map on a von-Neumann algebra $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$. Then there exists a family of linearly independent operators $\{v_\alpha : \alpha \in \mathcal{I}_\tau\}$ over the coefficients in $\mathcal{M}'$, commutant of $\mathcal{M}$ such that $\sum_{\alpha \in \mathcal{I}_\tau} v_\alpha v_\alpha^* = I$ and $\tau(x) = \sum_\alpha v_\alpha x v_\alpha^*$ for all $x \in \mathcal{M}$. If $\{v'_\beta : \beta \in \mathcal{J}_\tau\}$ be another such a family of operators representing τ, then there exists a family of elements $\{w^\alpha_\beta \in \mathcal{M}' : \alpha \in \mathcal{I}_\tau,\ \beta \in \mathcal{J}_\tau\}$ such that $v'_\beta = \sum_\alpha v_\alpha w^\alpha_\beta$ and $\sum_\beta w^\alpha_\beta (w^{\alpha'}_\beta)^* = \delta^\alpha_{\alpha'}$. τ is an extremal element in CP_σ if and only if there exists no non-trivial solution with elements $\lambda^\alpha_\beta \in \mathcal{M}'$ satisfying $\sum_{\alpha,\beta \in \mathcal{I}_\tau} v_\alpha \lambda^\alpha_\beta v_\beta = 0$.

Further if $\tau : \mathcal{M} \to \mathcal{M}$ and admits an inner representation i.e. with $v_\alpha \in \mathcal{M}$ then τ is extremal in the set of completely positive unital map on $\mathcal{M}$ if and only if there is no non-trivial solution (λ^α_β) in the center of $\mathcal{M}$.

PROOF: τ being normal and unital, π is a normal unital representation of $\mathcal{M}$ into $\mathcal{B}(\mathcal{K})$ and thus we may write $\mathcal{K} \equiv \mathcal{H} \otimes \mathcal{K}_0$ for some Hilbert space $\mathcal{K}_0$ and $\pi(x) \equiv x \otimes I_{\mathcal{K}_0}$. Thus we conclude first part by fixing an orthonormal basis e_α for $\mathcal{K}_0$ and defining $< f, v^*_\alpha g > = < f \otimes e_\alpha, V^*g >$. To show linear independence, let $\sum_\alpha c_\alpha v^*_\alpha = 0$ with $\sum |c_\alpha|^2 < \infty$. Then $f \otimes \sum c_\alpha e_\alpha$ is orthogonal to the total set of the vectors $\{(x \otimes I_{\mathcal{K}_0})V^*g : g \in \mathcal{H}, x \in \mathcal{M}\}$ in $\mathcal{K}$ and thus we get $f \otimes \sum_k c_\alpha e_\alpha = 0$. Thus $\sum_k c_\alpha e_\alpha = 0$ and (e_α) being linearly independent we get $c_\alpha = 0$. In fact same proof works to show for minimal representation the family v_α is linearly independent

over coefficients in $\mathcal{M}'$ i.e. $\sum_\alpha c_\alpha v_\alpha^* = 0$ with $c_\alpha \in \mathcal{M}'$ and $\sum_\alpha c_\alpha^* c_\alpha \in \mathcal{M}'$ if and only if $c_\alpha = 0$.

Choosing another such elements v'_β means choosing another basis for $\mathcal{K}_0$ and thus isometry property of W is obvious.

We give now the proof of the last two statements. First statement is a simple corollary of Theorem 2.8 and normality of τ. For the last statement, we compute with any unitary element $u \in \mathcal{M}'$ by uniqueness of the Radon-Nykodym representation of η for which $\eta \leq k\tau$ on $\mathcal{M}_+$ with a scaler $k > 0$, $\eta(x) = u\eta(x)u^* = u \sum_k v_k x t^k_j v_j^* u^* = \sum_k v_k x u t^k_j u^* v_j^*$ and thus by uniqueness of $t = (t^k_j)$ with entries in $\mathcal{M}'$, we get $ut^k_j u^* = t^k_j$ and thus matrix entries in $t = (t^k_j)$ are elements in $\mathcal{M} \bigcap \mathcal{M}'$. This completes the proof. ■

Once $\mathcal{A} = \mathcal{M}$ a von-Neumann algebra and τ is a normal map then the minimal Stinespring representation $\pi : \mathcal{M} \to \mathcal{B}(\mathcal{K})$ is also normal and thus $\mathcal{K} \equiv \mathcal{H} \otimes \mathcal{K}_0$ for a complex Hilbert space $\mathcal{K}$ so that $\pi(x) \equiv x \otimes I_{\mathcal{K}_0} E$ (See Section 2.4 in [BR1]) where E is a projection in the commutant of the range of the representation $x \to x \otimes I_{\mathcal{K}_0}$. Thus for an element $\tau \in CP^\sigma$ we have $\tau(x) = V(x \otimes I_{\mathcal{K}_0})V^*$ where $VV^* = I$ since τ is unital and so is π. So by fixing an orthonormal basis (e_i) for $\mathcal{K}_0$ we write $< f, v_k^* g > = < f \otimes e_k, V^* g >$ for all $f, g \in \mathcal{H}$ so that $V^* = \oplus_k v_k^*$ for a family of contraction $v_k^* \in \mathcal{B}(\mathcal{H})$ where index set could be countable or uncountable and

$$\tau(x) = \sum_k v_k x v_k^*, \ \forall x \in \mathcal{M}$$

where $\sum_k v_k v_k^* = 1$. Further the family $v_k^* : k \geq 1$ can be made linearly independent i.e. we can take a minimal possible choice for the Hilbert space $\mathcal{K}_0$. Explicitly if there exists a non-zero unit vector (λ_k) i.e. $\lambda_k \in \mathcal{M}'$ and $\sum_k \lambda_k^* \lambda_k = I$ such that $\sum_k \lambda_k v_k^* = 0$, we can set $w_k = \sum_j \lambda^k_j v_j^*$ for $k \geq 1$ where (w^i_j) is a unitary matrix taking values in $\mathcal{M}'$ with first row as $\lambda^1_k = \lambda_k$. Then we check that $\tau(x) = \sum_{k \geq 1} w_k x w_k^*$ where $w_1 = 0$. The minimal Stinespring Hilbert space being determined upto unitary equivalence, dimension of minimal choice for $\mathcal{K}$ is an invariance for τ i.e. will remain same under

cocycle conjugacy. We call the minimal possible dimension of $\mathcal{K}$ as Arveson's index for τ. It is evident that the index is an element in $\aleph_0$ if $\mathcal{H}$ is separable.

However for a given element $\tau \in CP^{\sigma}$, such a choice for a family of linearly independent contractions $\{v_k^* : k \geq 1\}$ representing τ is not unique. However if $\mathcal{M} = \mathcal{B}(\mathcal{H})$, the vector space $\mathcal{L}_\tau$ generated by $\{v_k : k \geq 1\}$ is determined uniquely by τ [Ar2]. Further we can give an inner product $<< .,. >>$ on $\mathcal{L}_\tau$ and choose v_k such that $<< v_k, v_j >>= 0$ for $j \neq k$. The dimension of the vector space $\mathcal{L}_\tau$ is called in the literature as Arveson index for τ. Thus our notion of Arveson index indeed is a generalization of Arveson index for a unital element τ in $CP^{\sigma}(\mathcal{B}(\mathcal{H}))$.

Given a unital C^*-algebra $\mathcal{A}$, a subspace $\mathcal{S}$ of $\mathcal{A}$ is called operator system if $\mathcal{S}$ is closed under involution $*$ and identity of $\mathcal{A}$ denoted by $I \in \mathcal{S}$. Let $\mathcal{S}_1, \mathcal{S}_2$ be two operator systems in $\mathcal{A}$. A linear one to one and onto map $\mathcal{I} : \mathcal{S}_1 \rightarrow \mathcal{S}_2$ is called order isomorphism if $\mathcal{I}$ and $\mathcal{I}^{-1}$ are both non-negative i.e. taking non-negative elements to non-negative elements. It is called a complete order isomorphism if $\mathcal{I} \otimes I_n : M_n(\mathcal{S}_1) \rightarrow M_n(\mathcal{S}_2)$ is also an order-isomorphism for each $n \geq 1$.

Let $\mathcal{M}$ be a von-Neumann algebra. We set notation $\mathcal{S}^\tau$ for the operator system

$$\mathcal{S}_v^\tau = \{\sum v_i \lambda_j^i v_j^* : \lambda_j^i \in \mathcal{M}'\}$$

for a unital completely positive normal map τ on $\mathcal{M}$ with Stinespring minimal representation $\tau(x) = \sum_k v_k x v_k^*$ for all $x \in \mathcal{M}$. One natural question that arises here: does operator space $\mathcal{S}_v^\tau$ depends on the choice that we make for (v_k) to represent τ? It is clear by Corollary 2.9 that that $\mathcal{S}^\tau$ is independent of choice that we make. The following proposition says now little more.

PROPOSITION 2.10: Let τ be an element in CP^{σ} then $\mathcal{S}^\tau$ is determined uniquely modulo an unitary operator $u \in \mathcal{A}'$ i.e. if $\tau(x) = \sum_k v_k x v_k^*$ and $\tau(x) = \sum_k w_k x w_k^*$ be two minimal Stinespring representation of τ then there exists an unitary operator $U = ((u_j^i)) \in M_n(\mathcal{M}')$ so that $UV^* = W^*$ and so operator system $\mathcal{S}^\tau$ is uniquely determined by τ with it's minimal representation i.e. there exists an unitary matrix

$\lambda = (\lambda^i_j)$ with entries in $\mathcal{M}'$ of order equal to numerical index so that

$$v^*_k = \sum_j \lambda^k_j w^*_j \tag{2.4}$$

Conversely let $\tau(x) = \sum_k v_k x v^*_k$ be a minimal representation of an extremal element τ in CP_σ and $\eta = \sum_k w_k x w^*_k$ be another minimal representation of an element $\eta \in CP_\sigma$ so that (w^*_k) is in the linear span of (v^*_k) over $\mathcal{M}'$ and $\mathcal{S}^\tau = \mathcal{S}^\eta$ then $\eta = \tau$.

PROOF: By the uniqueness part of Stinespring intertwining relation find an unitary operator U on $\mathcal{H} \otimes \mathcal{K}$ so that relation we have $U(x \otimes I)U^* = x \otimes I$ and $UW^* = V^*$. Thus we get in the basis $v^*_k = \sum_j u^k_j w^*_j$ where $U = (u^j_k)$ unitary elements in $M_n(\mathcal{M}')$ determined by $< f \otimes e_i, Ug \otimes e_j >=< f, u^i_j g >$ where n is the cardinality of an orthonormal basis for $\mathcal{K}$. Now we consider the relation $UW^*WU^* = V^*V$ where $U \in M_n(\mathcal{M}')$. Thus change of the basis for $\mathcal{K}$ will not change the operator spaces $\mathcal{S}^\tau_v$ and $\mathcal{S}^\tau_w$.

For the last part we fix $\lambda^i_j \in \mathcal{M}'$ so that $W^* = \Lambda V^*$ and claim that Λ is an isometry if τ is an extremal element. To that end we check $I = WW^* = V\Lambda^*\Lambda V^*$ i.e. $V(I - \Lambda^*\Lambda)V^* = 0$. τ being extremal we get $\Lambda^*\Lambda - I = 0$ by Corollary 2.9. Since $\lambda^i_j \in \mathcal{M}'$, we get by a simple computation that for all $x \in \mathcal{M}$,

$$\eta(x) = \sum w_k x w^*_k = W(x \otimes I)W^* = V\Lambda(x \otimes I)\Lambda^* V^* = V(x \otimes I)V^* = \sum_k v_k x v^*_k = \tau(x)$$

i.e. $\eta = \tau$. ∎

The set of unital completely positive map CP on $\mathcal{M}$ is compact in Arveson's BW(bounded weak) topology [Ar1,Pa Chapter 7] but normal unital completely positive maps CP_σ on $\mathcal{M}$ need not be a closed subset in BW topology and thus need not be compact in general in BW topology. For general facts on BW topology we refer readers to Chapter 7 in [Pa]. However we have the following simple observation.

PROPOSITION 2.11 The open dense convex set CP_σ in CP with respect to BW

topology is also a face of CP.

PROOF: We have already proved open dense property in Theorem 2.5. Let $\tau \in CP_\sigma$ and $\tau_0, \tau_1 \in CP$ so that $\tau = \lambda\tau_1 + (1-\lambda)\tau_0$ for some $\lambda \in (0,1)$. Then we have $\tau_1 \le \frac{1}{\lambda}\tau$ and also $\tau_0 \le \frac{1}{1-\lambda}\tau$ on non-negative elements $\mathcal{M}_+$. Normal property of a positive map η on $\mathcal{M}$ is equivalent to the property that the map η takes a decreasing net $x_\alpha \downarrow 0$ in $\mathcal{M}$ to a decreasing net $\eta(x_\alpha) \downarrow 0$. Thus normality of τ_0, τ_1 follows from that of τ. ■

Given an element $\tau \in CP_\sigma$ we consider the set

$$CP^\tau = \{\eta : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) : \text{completely positive and } \eta_| \mathcal{M} = \tau\}$$

Note that an element $\eta \in CP^\tau$ meed not be normal by our definition. That CP^τ is a closed convex subset in BW topology on the set of unital completely positive maps on $\mathcal{B}(\mathcal{H})$ is clear. Thus CP^τ is compact in BW topology. That the set CP^τ is non-empty follows by Arveson's Hann-Banach extension theorem which does not require τ to be normal and that it contains a normal element also follows from extending trivially Stinespring representation of the normal map $\tau(x) = \sum_k v_k x v_k^*$ by plugging in arbitrary $x \in \mathcal{B}(\mathcal{H})$. Any such an extension is an extremal element if τ is an extremal element in CP_σ. For a proof we can use Corollary 2.10. We denote by CP^τ_σ for normal elements in CP^τ. Thus CP^τ_σ is a non-empty convex face in CP^τ.

Question that we may ask now for an extremal element $\tau \in CP_\sigma$:

(a) What are it's extremal elements $\eta \in CP^\tau_\sigma$?

(b) Does an extremal element $\eta \in CP^\tau$ is also extremal in the convex set of unital CP maps on $\mathcal{B}(\mathcal{H})$?

The following theorem gives an answer to this question precisely.

THEOREM 2.12: Let τ be an extremal element the convex set of normal unital completely positive maps on a von-Neumann algebra $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$. Then CP^τ_σ is a non-empty convex set which is a face in the set CP of unital

completely positive normal maps on $\mathcal{B}(\mathcal{H})$. Any extremal element in CP^{τ}_{σ} is a natural extension of τ with index equal to that of τ.

PROOF: Let $\eta \in CP^{\tau}_{\sigma}$ and $\eta_0, \eta_1 \in CP$ so that $\eta = \lambda\eta_1 + (1-\lambda)\eta_0$. That η_0, η_1 are normal follows by Proposition 2.11. Taking restrictions of η_0, η_1 to $\mathcal{M}$ we get $\tau = \lambda\eta_1 + (1-\lambda)\eta_0$ on $\mathcal{M}$ where η_k are CP maps from $\mathcal{M}$ to $\mathcal{B}(\mathcal{H})$. Since τ is also extremal in the convex set of completely positive maps from $\mathcal{M}$ to $\mathcal{B}(\mathcal{H})$ as abstract criteria obtained in Proposition 2.9 remains same if we treat τ as an element in the larger convex set. Thus $\eta_k = \tau$ on $\mathcal{M}$ for $k = 0, 1$. This shows that CP^{τ}_{σ} is a face in the set of all unital completely positive maps on $\mathcal{B}(\mathcal{H})$. Thus any extremal element in CP^{τ}_{σ} is also an extremal element in the set of all unital completely positive map on $\mathcal{B}(\mathcal{H})$. ■

3 Tomita's coupling and extremal marginal states:

We now aim to describe extremal marginal states in a more general mathematical frame work then originally proposed [Par] and followed worked [PSa,Ru,Oh]. Let ϕ be a faithful normal state on a von-Neumann algebra $\mathcal{M}$ acting on a complex separable Hilbert space $\mathcal{H}$ with inner product $< .,. >$ taken linear in the second variable and without loss of generality we assume $\mathcal{M}$ be in it's standard form $(\mathcal{M}, \mathcal{P}, \mathcal{J}, \Delta, \Omega)$ [BR] associated with Ω where Ω is a cyclic and separating vector for $\mathcal{M}$ and $\Delta, \mathcal{J}$ are Tomita's modular and conjugate operator associated with $S = \mathcal{J}\Delta^{\frac{1}{2}}$, which is the closer of of the densely defined anti-linear closeable operator $S_0 x\Omega \rightarrow x^*\Omega$ and $\mathcal{P} = \overline{\{x\mathcal{J}x\mathcal{J}\Omega : x \in \mathcal{M}\}}$ is the self-dual pointed positive cone in $\mathcal{H}$. Here we recall that analytic elements $\mathcal{M}_a$ of the modular group $\sigma_t(x) = \Delta^{it}x\Delta^{-it}$ on $\mathcal{M}$ is dense in weak* topology in $\mathcal{M}$ and we have the following modular relation for any two $x, y \in \mathcal{M}_a$ given by

$$\phi(x^*y) = \phi(\sigma_{\frac{i}{2}}(y)\sigma_{-\frac{i}{2}}(x^*)) \tag{3.1}$$

We consider the set of states ψ on C* tensor product of von-Neumann algebras

$\mathcal{M} \circ \mathcal{M}$ acting on $\mathcal{H} \otimes \mathcal{H}$ such that it's restrictions on $\mathcal{M} \circ I$ and $I \circ \mathcal{M}$ are equal to ϕ. In short we will call such a state a coupling state with marginal ϕ and denote the convex set by C_ϕ. Simplest example is the product state and simplest non-product state is given by a Tomita state

$$\psi(x \otimes y) =< \mathcal{J} x \mathcal{J} \Omega, y \Omega >$$

by extending linearly and then to it's norm closures $\mathcal{M} \circ \mathcal{M}$. Important difference that we note now that ψ may not have a normal extension to von-Neumann completion of $\mathcal{M} \circ \mathcal{M}$ i.e. to $\mathcal{M} \otimes \mathcal{M} = (\mathcal{M} \circ \mathcal{M})''$. As an example we take $\mathcal{M} = L^\infty[0,1]$ and check that indicator function of the diagonal set in $[0,1] \times [0,1]$ can be expressed as limit of decreasing projections $E_n \in \mathcal{M} \circ \mathcal{M}$ with $\psi(E_n) = 1$ and $\bigcap_{n \geq 1} E_n = \{[x,x] : 0 \leq x \leq 1\}$. In-spite of this odd feature we have a simple but beautiful observation.

LEMMA 3.1: A state $\psi \in C_\phi$ if and only if there exists a unital normal map $\tau_\psi : \mathcal{M} \to \mathcal{M}$ preserving ϕ so that

$$\psi(x \otimes y) =< \mathcal{J} x \mathcal{J} \Omega, \tau_\psi(y) \Omega > \tag{3.2}$$

for all $x, y \in \mathcal{M}$. Further the map $\psi \to \tau_\psi$ is an one to one and onto affine map between two convex set C_ϕ and

$$CP_\phi = \{\tau : \mathcal{M} \to \mathcal{M}, \text{ completely positive unital map and } \phi \circ \tau = \phi\}$$

PROOF: We fix $y \geq 0$ so that $\phi(y) = 1$ and consider that state $\phi_y : x \to \psi(x \otimes y)$ and note that $\phi_y(x) \leq ||y|| \phi(x)$ for $x \geq 0$ as $0 \leq y \leq ||y|| I$. Since by our assumption ϕ is normal, ϕ_y is also normal. Thus by Dixmier's lemma we conclude that $\phi_y(x) =< y'^* \Omega, x \Omega >$ for some non-negative element $y' \in \mathcal{M}'$. Thus we get $\psi(x \otimes y) =< \mathcal{J} x \mathcal{J} \Omega, \mathcal{J} y'^* \mathcal{J} \Omega >$. We set $\tau(y) = \mathcal{J} y'^* \mathcal{J} \in \mathcal{M}$ to arrive at (3.2). Since y' is determined uniquely by separating property of $\mathcal{M}'$, we may extend the map for an arbitrary element y using linearity by writing it as a linear combination of four non negative elements in $\mathcal{M}$. That $y \to \tau(y)$ is a normal map follows by invariance of faithful normal state ϕ for τ and positivity of the map as follows: for an increasing

net y_α with least upper bound y, $\tau(y_\alpha)$ is also an increasing net with $\tau(y)$ as an upper bound and thus has a least upper bound say z. Then $\phi(z-\tau(y)) = \text{l.u.b.}_\alpha \phi(\tau(y_\alpha) - \tau(y))$ by normality of ϕ and thus by invariance equal to $\text{l.u.b.}_\alpha \phi(y_\alpha - y)$ which is 0 by normality of the state. For complete positive property of the map τ, we first check that $\phi(\sum_{i,j} z'_i \tau(y_i y_j^*) z'^*_j) = \psi(\sum_{i,j} z_i z_j^* \otimes y_i y_j^*) = \psi(X^* X) \geq 0$ where $X = \sum_i z_i \otimes y_i$ for all $1 \leq i \leq n$, $y_i \in \mathcal{M}, z'_i \in \mathcal{M}'$ and $z_i = \mathcal{J} z'_i \mathcal{J}$. Since Ω is cyclic for $\mathcal{M}'$, we conclude that the operator $((\tau(y_i y_j^*)))$ is a non-negative element in $M_n(\mathcal{M})$. Thus τ is $n-$positive for each $n \geq 1$. Rest of the statements are now obvious. ■

THEOREM 3.2: The affine map $\psi \to \tau_\psi$ defined in Lemma 3.1 takes extremal elements of the convex set C_ϕ to extremal elements of CP_ϕ. Further

(a) C_ϕ is a closed convex subset in the weak topology of $\mathcal{M} \circ \mathcal{M}$;

(b) CP_ϕ is also compact once equipped with the point-wise topology inherited from σ-weak operator topology of $\mathcal{M}$ (i.e. we say a net $\tau_\alpha \in CP$ converges to $\tau \in CP$ if the net $\tau_\alpha(x)$ converges to $\tau(x)$ in $\sigma-$weak operator topology for each $x \in \mathcal{M}$, where CP denotes the set of unital completely positive maps on $\mathcal{M}$).

PROOF: First part is obvious as the map is one to one and onto. Crucial observation that we make here for a net τ_α in CP_ϕ such that $\tau_\alpha(x) \to \tau(x)$ in $\sigma-$weak operator topology for some $\tau \in CP$. Then τ is also unital and ϕ preserving and τ is normal as ϕ is faithful. Thus $\tau \in C_\phi$.

We consider a net of states $\psi_\alpha \in C_\phi$ so that $\psi_\alpha \to \psi$ in weak topology on $\mathcal{M} \circ \mathcal{M}$ and since each ψ_α preserves marginals so is their limit. Thus $\psi \in C_\phi$. Now we also check that $\psi_\alpha(x \otimes y) = \phi(\mathcal{J} x \mathcal{J} \Omega, \tau_\alpha(y)\Omega > \to \psi(x \otimes y)$ for all $x, y \in \mathcal{M}$ where the state ψ on $\mathcal{M} \circ \mathcal{M}$ defined by $\psi(x \otimes y) =< \mathcal{J} x \mathcal{J} \Omega, \tau(y)\Omega >$ determines a unique element $\tau \in CP_\phi$ by Lemma 3.1. These shows that $\tau_\alpha(x) \to \tau(x)$ in weak operator topology. Since the family is uniformly bounded, the limit also holds in $\sigma-$weak operator topology by dominated convergence theorem. Conversely for a given net τ_α in CP_ϕ which converges to τ in point-wise $\sigma-$weak operator topology, $\tau \in CP_\phi$ by our remark at the beginning of the proof. We check that the associated elements ψ_α

also converges to ψ in the weak topology of $\mathcal{M} \circ \mathcal{M}$ first on a norm dense subspace of algebraic tensor product of $\mathcal{M} \underline{\otimes} \mathcal{M}$ and then for any arbitrary elements by a standard argument as the net is uniformly bounded.

That CP_ϕ is compact follows from compactness of C_ϕ being a close subset of a compact set and above correspondence which respect the topologies. ■

One case as well use Arveson's BW (Bounded Weak) topology [Ar1,Pa, Chapter 7] directly to give a direct proof that CP_ϕ is a closed subset of the unit ball of bounded linear maps on $\mathcal{M}$. Any limit points of a convergent net of ϕ-invariant unital completely positive normal maps τ_α will be also completely positive and ϕ-invariant. ϕ being faithful and normal, any positive ϕ invariant map is also normal. However such an argument will not be valid with BW topology for a more general situation where ϕ is just a normal $\sigma-$finite weight [BR1,Ta]. It is not clear what would be a possible truncation method along the classic work [Ke,Ko].

By our last Theorem we conclude that extremal elements in CP_ϕ exists and any element in CP_ϕ admits Krein-Milman's property. In the following text we aim to find a criteria for an element $\tau \in CP_\phi$ to be normal. To that end we recall adjoint completely positive map [OP].

PROPOSITION 3.3: Let ϕ be a faithful normal state on a von-Neumann algebra $\mathcal{M}$ and τ be an element in CP_ϕ. Then there exists a unique element $\tilde{\tau} \in CP_\phi$ satisfying the following duality relation

$$\phi(\tau(x)\sigma_{-\frac{i}{2}}(y)) = \phi(\sigma_{\frac{i}{2}}(x)\tilde{\tau}(y)) \tag{3.3}$$

where $x, y \in \mathcal{M}_a$, the $*$-algebra of analytic elements for Tomita's modular group $\sigma = (\sigma_t : t \in I\!R)$ associated with ϕ. Further numerical indices of τ and $\tilde{\tau}$ are equal. If τ admits an inner Stinespring minimal representation

$$\tau(x) = \sum_{\alpha \in \mathcal{C}} v_\alpha x v_\alpha^*, \; x \in \mathcal{M}$$

i.e. with elements $v_\alpha \in \mathcal{M}$ then $\tilde{\tau}$ also admits an inner Stinespring minimal representation given by and

$$\tilde{\tau}(y) = \sum_{\alpha \in \mathcal{C}} \tilde{v}_\alpha y \tilde{v}_\alpha^*, \ y \in \mathcal{M}$$

where $\tilde{v}_\alpha \in \mathcal{M}$ is defined as the bounded operator extending the densely defined operator $\Delta^{\frac{1}{2}} v_\alpha^* \Delta^{-\frac{1}{2}}$ [BJKW].

PROOF: For the 1st part of the statement we refer to chapter 8 of monograph [OP]. For the second part we re-investigate the proof of Stinespring representation with our special situation. With out loss of generality we assume that $\mathcal{M}$ is in the standard form associated with ϕ. i.e. $\phi(x) = (\Omega, x\Omega)$ where Ω is a cyclic and separating vector for $\mathcal{H}$. We set kernel on $\mathcal{M}_a \otimes \mathcal{M}_a$ defined by

$$k(x \otimes z, y \otimes w) = < \Omega, x^* \tau(z^* w) y \Omega >$$

That Hilbert space completion of the kernel is same as that of Stinespring follows by cyclic property of Ω. Similarly we set kernel $\tilde{k}$ associated with $\tilde{\tau}$ by

$$\tilde{k}(x \otimes z, y \otimes w) = < \Omega, z^* \tau(x^* y) w \Omega >$$

in reverse direction and check by KMS relation that

$$< \Omega, x^* \tau(z^* w) y \Omega > = < \Omega, \tilde{w}^* \tilde{\tau}(\tilde{y}^* \tilde{x}) \tilde{z} \Omega >$$

where for any analytic element $x \in \mathcal{M}$ we write $\tilde{x} = \sigma_{-\frac{i}{2}}(x^*)$. Such a relation is used already in [AM,Mo1]. This clearly shows that Hilbert spaces associated with kernel k and $\tilde{k}$ are conjugated by an anti-unitary operator defined by $U : x \otimes y \to \tilde{y} \otimes \tilde{x}$. Since anti-unitary operator U inter-twins the representations, it also inter-twins $\mathcal{K}_0$ and $\tilde{\mathcal{K}}_0$ and thus we get dimension of $\mathcal{K}_0$ and $\mathcal{K}_0'$ are same. Thus index of τ and $\tilde{\tau}$ are equal. For the last part we refer section 7 and appendix given in the article [BJKW]. ■

In the following armed with last proposition, we aim to a generalization of Landau-Streater's criteria [LS] given when $\mathcal{M}$ is a matrix algebra and ϕ is the normalize trace. Note that our proof follows quite a different method as in Theorem 2.8.

THEOREM 3.4: Let τ be an element in CP_ϕ admitting an inner minimal Stinespring representation $\tau(x) = \sum_k v_k x v_k^*,\ x \in \mathcal{M}$ and $\tilde{\tau}(x) = \sum \tilde{v}_k x \tilde{v}_k^*$ be the associated Stinespring's minimal inner representation of $\tilde{\tau}$ as described in the Proposition 3.3. Then τ is an extremal element in CP_ϕ if and only if there exists no non-trivial $\lambda = (\lambda_j^k)$ with entries in the center of $\mathcal{M}$ satisfying the relation

$$\sum v_k \lambda_j^k v_j^* = 0, \ \ \sum \tilde{v}_j \lambda_j^k \tilde{v}_k^* = 0 \tag{3.4}$$

PROOF: Let η be an element in CP_ϕ such that $\eta \leq k\tau$ for some $k > 0$. Then there exists a non-negative element $t = (t_\beta^\alpha)$ with entries in the center of $\mathcal{M}$ such that $\eta(x) = \sum v_\alpha x t_\beta^\alpha v_\beta^* = \sum w_\alpha x w_\alpha^*$ for all $x \in \mathcal{M}$ where $W^* = \mu V^*$ where $\mu = (\mu_\beta^\alpha)$ is self-adjoint matrix with entries in the center of $\mathcal{M}$ satisfying $\mu^2 = t$. In particular η is also inner and thus $\tilde{\eta}$ also admits inner minimal representation

$$\tilde{\eta}(x) = \sum \tilde{w}_\alpha x \tilde{w}_\alpha^*$$

given in Proposition 3.3.

Since $W^* = \mu V^*$, a simple computation using modular relation (3.1) now also leads to the minimal representation

$$\tilde{\eta}(x) = \sum \tilde{v}_\alpha x t_\beta^\alpha \tilde{v}_\beta^*$$

since modular group acts trivially on the center of $\mathcal{M}$.

Let η_0, η be two elements in CP_ϕ such that $\tau = \lambda\eta + (1-\lambda)\eta_0$ for some $\lambda \in (0,1)$. Then we also have by the affine map $\tau \to \tilde{\tau}$ on CP_ϕ given in Proposition 3.2 that $\tilde{\tau} = \lambda\tilde{\eta} + (1-\lambda)\tilde{\eta}_0$. Thus $\lambda_\beta^\alpha = t_\beta^\alpha - I$ is a solution to (3.4) and it is non-trivial if and only if τ is non an extremal element in CP_ϕ. This completes the proof. ■

One may expect a general result dropping our assumption of inner property and proof seems not so automatic. Krein-Milman theorem says that $\tau = \int_{\overline{CP_\phi^e}} eta\mu(e)$ for some probability measure μ on the closer of extreme points of CP_ϕ where μ may

not be uniquely determined. A valid question that rises here when can we guarantee μ to have support on CP^e_ϕ only?

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